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# Bottleneck options

**Curdin Ott**

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**Abstract** In the spirit of Kyprianou and Ott (in Acta Appl. Math., to appear, 2013) and Ott (in Ann. Appl. Probab. 23:2327–2356, 2013) we consider an option whose payoff corresponds to a capped American lookback option with floating strike and solve the associated pricing problem (an optimal stopping problem) in a financial market whose price process is modelled by an exponential spectrally negative Lévy process. Despite the simple interpretation of the cap as a moderation of the payoff, it turns out that the optimal strategy to exercise the option looks very different compared to the situation without a cap. In fact, we show that the continuation region has a feature that resembles a bottleneck and hence the name "bottleneck option".

**Keywords** Bottleneck option · Optimal stopping · Principle of smooth and continuous fit · Lévy processes · Scale functions

**Mathematics Subject Classification** 60G40 · 60G51 · 60J75

**JEL Classification** G13

## 1 Introduction

Consider a financial market consisting of a riskless bond and a risky asset whose price is modelled by a positive stochastic process $S = \{S_t : t \geq 0\}$. A "bottleneck option" (the name will be justified in due course) gives the holder the right to exercise at any finite time $\tau$ (a stopping time) yielding payouts

$$e^{-\alpha\tau}\left(M_0 \vee \left(\sup_{0\leq u\leq\tau} S_u \wedge C\right) - K S_\tau\right)^+, \quad C > M_0 \geq S_0, \alpha \geq 0. \tag{1.1}$$

C. Ott (✉)
RiskLab, Department of Mathematics, ETH Zurich, Rämistrasse 101, 8092 Zurich, Switzerland
e-mail: curdin.ott@math.ethz.ch

The constant $M_0$ can be viewed as representing the "starting" maximum of the stock price (say, over some previous period $(-t_0, 0])$, $K > 0$ is referred to as strike, $\alpha$ is a discount factor and $C$ is the cap. This type of payoff belongs to the class of so-called perpetual "lookback" options—"lookback" because it involves the term $\sup_{0 \le u \le \tau} S_u$ and thus the holder of such an option has to look back in time in order to determine the payoff at time $\tau$. The simplest example is a Russian option which was introduced by Shepp and Shiryaev [22, 23] and corresponds to setting $K = 0$ and $C = \infty$ above. Another example would be an American lookback option with fixed strike which is (1.1) with $C = \infty$ and the term $K S_\tau$ replaced by $K$; cf. [6, 7, 16].

Assuming that $C = \infty$ and taking into account the particular form of the payoff in (1.1), one sees that it is positive at time $t$ provided the quantity $\overline{S}_t - S_t$ is sufficiently large, where $\overline{S} = \{\overline{S}_u : u \ge 0\}$ is given by $\overline{S}_u := M_0 \vee \sup_{0 \le v \le u} S_v$. We refer to the quantity $\overline{S}_t - S_t$ as the depth of the excursion of $S$ away from its running maximum. In view of the discounting in (1.1), this suggests that it is worth exercising the option as soon as $S$ undertakes an excursion away from its running maximum that is deep enough. Thus a payoff of the form (1.1) could be particularly interesting for an investor interested in exploiting instances when $S$ drops significantly after reaching new maxima. Payoffs of type (1.1) with $C = \infty$ have been studied before and are sometimes called American lookback options with floating strike; cf. [5, 6]. One additional feature here is that we allow $C < \infty$ which corresponds to a moderation of the payoff in the sense that it is bounded from above by $C$. We therefore refer to $C$ as the cap. The case when $C = \infty$ simply means no moderation at all. Alternatively, the cap can be viewed as a means to limit the downside risk for an issuer of a payoff of the form (1.1).

Apart from the simple economic interpretation of the cap mentioned in the previous paragraph, we show that its presence has a surprising effect on the optimal exercise strategy. Here optimal is understood in the sense that the expected discounted payoff is maximised. As informally described above, if $C = \infty$, it is plausible that the optimal strategy to exercise (1.1) is to wait until $S$ undertakes an excursion away from its running maximum that is deep enough. In fact, this was proved rigorously for a Black–Scholes model in [5, 16]. Their result can be visualised by drawing the trace of a realisation of the process $t \mapsto (S_t, \overline{S}_t)$ in the positive quadrant; see Fig. 1. The grey area corresponds to the continuation region, that is, the region where one continues to observe the evolution of $(S, \overline{S})$ and does not exercise the option. Note that the dynamics of $(S, \overline{S})$ are such that it can only climb upwards along the diagonal. The horizontal lines in Fig. 1 are meant to schematically indicate the trace of the excursions of $S$ away from its running maximum.

On the other hand, if $C < \infty$ and $K > 1$, we show that in a specific class of models, which includes the Black–Scholes model, the optimal strategy to exercise (1.1) is of the following form: As long as the second component of $(S_t, \overline{S}_t)$ lies below $C$, one waits until $S$ undergoes an excursion away from its running maximum of depth at least $g(\overline{S}_t)$ for some function $g$. Once the level $C$ is reached, the strategy consists of stopping as soon as $S_t$ drops below a fixed value. Pictorially displaying this (see Fig. 1), one sees that the continuation region has a feature that resembles a bottleneck and hence the name "bottleneck option". Furthermore, it turns out that as one decreases $K$, the bottleneck becomes smaller and smaller and eventually vanishes once

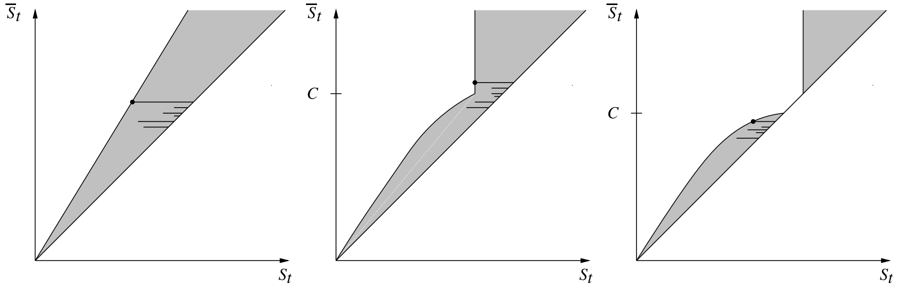


**Fig. 1** The anticipated continuation region (*grey*) and stopping region for the cases when $C = \infty$, $C < \infty$ and $K > 1$, and $C < \infty$ and $K$ small enough

$K$ drops below a critical value. The resulting continuation region then consists of two disjoint regions; see Fig. 1.

In order to make things more rigorous, let us specify the underlying model. Suppose that $X = \{X_t : t \geq 0\}$ is a spectrally negative Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$ satisfying the natural conditions; cf. [3], Sect. 1.3, page 39. For $x \in \mathbb{R}$, denote by $\mathbb{P}_x$ the probability measure under which $X$ starts at $x$ and for simplicity write $\mathbb{P}_0 = \mathbb{P}$. The value of the bond $B = \{B_t : t \geq 0\}$ evolves deterministically such that

$$B_t = B_0 e^{rt}, \quad B_0 > 0, r \geq 0, t \geq 0,$$

and the price of the risky asset is modelled as the exponential spectrally negative Lévy process

$$S_t = S_0 e^{X_t}, \quad S_0 > 0, t \geq 0.$$

If also dividends are taken into account, the capital $\tilde{S} = \{\tilde{S}_t : t \geq 0\}$ of the stockholder is assumed to evolve according to

$$\tilde{S}_t = e^{\delta t} S_t = S_0 e^{\delta t + X_t}, \quad t \geq 0,$$

where $\delta \geq 0$ is the rate at which dividends are paid. In order to guarantee that our model is free of arbitrage, we assume that $\psi(1) = r - \delta$, where $\psi$ is the Laplace exponent of $X$ under $\mathbb{P}$. Put differently, $e^{-rt}\tilde{S}_t$, $t \geq 0$, is a martingale under $\mathbb{P}$. If $X_t = \mu t + \sigma W_t$ where $W = \{W_t : t \geq 0\}$ is a standard Brownian motion, we get the standard Black–Scholes model for the price of the asset. Of course, it is an important question whether this model of a financial market is appropriate, but we do not discuss this issue here. Nevertheless, it is worth mentioning that Carr and Wu [4] as well as Madan and Schoutens [12] offer empirical evidence for appropriate market scenarios to support this model in which the risky asset is driven by a spectrally negative Lévy process.

Finding the optimal time to exercise (1.1) and the corresponding expected payoff leads by the standard theory of pricing American-type options [24] to solving the optimal stopping problem

$$V_r(M_0, S_0, C) := B_0 \sup_\tau \mathbb{E}\left[B_\tau^{-1} e^{-\alpha\tau}\left(M_0 \vee \left(\sup_{0\le u\le\tau} S_u \wedge C\right) - K S_\tau\right)^+\right], \quad (1.2)$$

where the supremum is taken over all $[0, \infty)$-valued stopping times. In other words, we want to find a stopping time which optimizes the expected discounted payoff. It will be convenient to rewrite (1.2) in a slightly different way. Specifically, we associate with $X$ the maximum process $\overline{X} = \{\overline{X}_t : t \ge 0\}$, where $\overline{X}_t := s \vee \sup_{0\le u\le t} X_u$ for $t \ge 0$, $s \ge x$. The law under which $(X, \overline{X})$ starts at $(x, s)$ is denoted by $\mathbb{P}_{x,s}$. Thus, summing up, the aim of this article is to solve the optimal stopping problem

$$V_\epsilon^*(x, s) = \sup_{\tau\in\mathcal{M}} \mathbb{E}_{x,s}\left[e^{-q\tau}\left(e^{\overline{X}_\tau\wedge\epsilon} - K e^{X_\tau}\right)^+\right], \quad (1.3)$$

where $q > 0$, $K > 0$, $\epsilon \in \mathbb{R} \cup \{\infty\}$, $(x, s) \in E := \{(x_1, s_1) \in \mathbb{R}^2 \mid x_1 \le s_1\}$ and $\mathcal{M}$ is the set of all $[0, \infty)$-valued $\mathbb{F}$-stopping times. In particular, note that we have $V_r(M_0, S_0, C) = V_\epsilon^*(x, s)$ with $x = \log S_0$, $s = \log M_0$, $\epsilon = \log C$, $q = \alpha + r$ and $\psi(1) = r - \delta$. When $\epsilon = \infty$, this problem was solved in [5, 16] for the case when $X$ is a linear Brownian motion and in [6] for the case when $X$ is a jump-diffusion. In the case when $\epsilon = \infty$ and $K = 0$, this problem is known as the Russian optimal stopping problem [2, 6, 22, 23]. Furthermore, if $\epsilon \in \mathbb{R} \cup \{\infty\}$ and $K = 0$, then the problem was considered in [15]. In fact, the technique used in [15] also plays a major role in this article as will become clear in due course. The connection between (1.3) and [15] is further discussed in Sect. 4 (see Remark 4.6). A slight modification of (1.3), namely when $\epsilon \in \mathbb{R} \cup \{\infty\}$ and the term $K e^{X_\tau}$ is replaced by $K$ in (1.3), has been studied in [11]. Finally, an optimal stopping problem that combines all the previously mentioned cases when $\epsilon = \infty$ and when $X$ is a linear Brownian motion has been investigated in [8].

Our method for solving (1.3) consists of a classical verification technique, that is, we heuristically derive a candidate solution and then verify that it is indeed a solution. As will become clear in due course (see Sects. 3 and 6), the candidate solution consists of two parts. One part is obtained by applying the principle of smooth and continuous fit [14, 18, 19] in a very similar way to [11, 15], whereas the other part is obtained by appropriately linking (1.3) to the so-called McKean optimal stopping problem [1, 13]. As one would expect from the general theory of optimal stopping [19, 25], the optimal stopping time is the first entry time of the two-dimensional Markov process $(X, \overline{X})$ into a certain subset (the stopping region) of $E$. Interestingly, and as already alluded to above, it turns out that depending on the different parameters, the continuation region (the complement of the stopping region) is a connected set or consists of two disjoint components. In fact, in the former case it has a feature that resembles a bottleneck; see Theorem 4.4 and Fig. 4. Furthermore, it will also be interesting from a technical point of view to see how the fact that the payoff depends not only on $\overline{X}$ but also on $X$ (compare with [11, 15] where the payoff only depends on $\overline{X}$) enters the solution of the optimal stopping problem.

One of the assumptions above is that the underlying Lévy process is spectrally negative, that is, a Lévy process whose trajectories have only negative discontinuities.

This restriction, which can be justified from a modelling point of view [4, 12], opens the door to the theory of scale functions for spectrally negative Lévy processes [9, 10] and essentially allows us to obtain the results in the form in which we are going to present them below. However, we believe that from a qualitative point of view the results should still hold even if one allows $X$ to be a general Lévy process. This would lead to interesting phenomena where the process $(S, \overline{S})$ jumps from one component of the continuation region to the other one in the case when the continuation region consists of two parts.

We conclude this section with a brief overview of this article. In Sect. 2 we introduce some more notation and provide some necessary background. In Sects. 3 and 6 we explain how to heuristically derive a candidate solution for (1.3). Our main results are presented in Sect. 4 and their proofs are given in Sect. 7. Finally, some examples are considered in Sect. 5.

## 2 Preliminaries

### 2.1 Spectrally negative Lévy processes

It is well known that a spectrally negative Lévy process $X$ is characterised by its Lévy triplet $(\gamma, \sigma, \Pi)$, where $\sigma \geq 0$, $\gamma \in \mathbb{R}$ and $\Pi$ is a measure on $(-\infty, 0)$ satisfying the condition $\int_{(-\infty,0)}(1 \wedge x^2)\,\Pi(dx) < \infty$. By the Lévy–Itô decomposition, the process may be represented in the form

$$X_t = \sigma B_t - \gamma t + X_t^{(1)} + X_t^{(2)},$$

where $\{B_t : t \geq 0\}$ is a standard Brownian motion, $\{X_t^{(1)} : t \geq 0\}$ is a compound Poisson process with discontinuities of magnitude bigger than or equal to one, $\{X_t^{(2)} : t \geq 0\}$ is a square-integrable martingale with discontinuities of magnitude strictly smaller than one, and the three processes are independent. In particular, if $X$ is of bounded variation, the decomposition reduces to

$$X_t = \mathtt{d}t - \chi_t$$

where $\mathtt{d} := -\gamma - \int_{(-1,0)} x\,\Pi(dx) > 0$ and $\{\chi_t : t \geq 0\}$ is a driftless subordinator. Further let

$$\psi(\theta) := \mathbb{E}\big[e^{\theta X_1}\big]$$

be the Laplace exponent of $X$ for all $\theta \in \mathbb{R}$ such that the expectation exists. Since $X$ is spectrally negative, this is at least the case for $\theta \geq 0$. It is known that $\psi$ takes the form

$$\psi(\theta) = -\gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} \big(e^{\theta x} - 1 - \theta x 1_{\{x>-1\}}\big)\,\Pi(dx), \quad \theta \geq 0.$$

When $X$ has bounded variation, that is, $\sigma = 0$ and $\int_{(-1,0)} |x|\,\Pi(dx) < \infty$, we may always write

$$\psi(\theta) = \mathtt{d}\theta - \int_{(-\infty,0)} (1 - e^{\theta x})\,\Pi(dx), \quad \theta \geq 0. \tag{2.1}$$

The right inverse of $\psi$ is defined by

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}$$

for $q \geq 0$.

For any spectrally negative Lévy process having $X_0 = 0$, we introduce the family of martingales

$$\exp\big(vX_t - \psi(v)t\big),$$

defined for any $v \in \mathbb{R}$ for which $\psi(v) < \infty$, and further the corresponding family of measures $\mathbb{P}^v$ with Radon–Nikodým derivatives

$$\frac{d\mathbb{P}^v}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \exp\big(vX_t - \psi(v)t\big). \tag{2.2}$$

For all such $v$, the measure $\mathbb{P}^v_x$ will denote the translation of $\mathbb{P}^v$ under which $X_0 = x$. In particular, under $\mathbb{P}^v_x$ the process $X$ is still a spectrally negative Lévy process; cf. Theorem 3.9 in [10].

Finally, introduce the first passage times of $X$ below and above $k \in \mathbb{R}$,

$$\tau_k^- = \inf\{t > 0 : X_t \leq k\} \quad \text{and} \quad \tau_k^+ = \inf\{t > 0 : X_t \geq k\}.$$

## 2.2 Scale functions

A special family of functions associated with spectrally negative Lévy processes is that of scale functions (cf. [9, 10]) which are defined as follows. For $q \geq 0$, the $q$-scale function $W^{(q)} : \mathbb{R} \longrightarrow [0, \infty)$ is the unique function whose restriction to $(0, \infty)$ is continuous and has Laplace transform

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)\, dx = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q),$$

and is defined to be identically zero for $x \leq 0$. Further, we shall use the notation $W_v^{(q)}(x)$ to mean the $q$-scale function associated to $X$ under $\mathbb{P}^v$. It is possible for fixed $x \geq 0$ to extend the mapping $q \mapsto W_v^{(q)}(x)$ to the complex plane (cf. Lemma 3.6 in [9]), and we have the relationship

$$W^{(q)}(x) = e^{vx} W_v^{(q-\psi(v))}(x) \tag{2.3}$$

for $v \in \mathbb{R}$ such that $\psi(v) < \infty$ and $q \in \mathbb{C}$; cf. Lemma 3.7 in [9]. Moreover, the following regularity properties of scale functions are known; cf. Sects. 2.3 and 3.1 of [9].

*Smoothness*: For all $q \geq 0$,

$$W^{(q)}\big|_{(0,\infty)} \in \begin{cases} C^1(0,\infty), & \text{if } X \text{ is of bounded variation and } \Pi \text{ is atomless}, \\ C^1(0,\infty), & \text{if } X \text{ is of unbounded variation and } \sigma = 0, \\ C^2(0,\infty), & \text{if } \sigma > 0. \end{cases}$$

*Continuity at the origin:* For all $q \geq 0$,

$$W^{(q)}(0+) = \begin{cases} \mathtt{d}^{-1}, & \text{if } X \text{ is of bounded variation,} \\ 0, & \text{if } X \text{ is of unbounded variation.} \end{cases} \tag{2.4}$$

*Right derivative at the origin:* For all $q \geq 0$,

$$W_+^{(q)\prime}(0+) = \begin{cases} \frac{q+\Pi(-\infty,0)}{\mathtt{d}^2}, & \text{if } \sigma = 0 \text{ and } \Pi(-\infty,0) < \infty, \\ \frac{2}{\sigma^2}, & \text{if } \sigma > 0 \text{ or } \Pi(-\infty,0) = \infty, \end{cases} \tag{2.5}$$

where we understand the second case to be $+\infty$ when $\sigma = 0$.

The second scale function is $Z_v^{(q)}$ which is defined as follows. For $v \in \mathbb{R}$ such that $\psi(v) < \infty$ and for $q \geq 0$, we define $Z_v^{(q)} : \mathbb{R} \longrightarrow [1, \infty)$ by

$$Z_v^{(q)}(x) = 1 + q \int_0^x W_v^{(q)}(z)\, dz. \tag{2.6}$$

This function can also be extended to $q \in \mathbb{C}$ for fixed $x \geq 0$.

For technical reasons, **we require for the rest of the paper that $W^{(q)}$ is in $C^1(0,\infty)$ [and hence $Z^{(q)} \in C^2(0,\infty)$]. This is ensured by henceforth assuming that $\Pi$ is atomless whenever $X$ is of bounded variation**.

## 3 First observations and candidate solution

The overall strategy to solve (1.3) is "guess and verify", that is, we first try to "guess" the solution of (1.3), and once we have a candidate solution we verify that it is indeed a solution. This section is concerned with the guessing part of our approach. We link (1.3) to the McKean optimal stopping problem (cf. [1, 13] and Sect. 9.2 of [10]) as well as to the general theory of optimally stopping a maximum process [17, 19] which will provide us with a candidate solution for (1.3). **Assume throughout this section that $\epsilon \in \mathbb{R}$.**

First of all, observe that if $s \geq \epsilon$, then $\overline{X}_t \wedge \epsilon$ equals $\epsilon$ for all $t \geq 0$ and (1.3) becomes

$$\begin{aligned} V_\epsilon^*(x,s) &= \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}\big[e^{-q\tau}\big(e^\epsilon - Ke^{X_\tau}\big)^+\big] \\ &= K \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}\big[e^{-q\tau}\big(K^{-1}e^\epsilon - e^{X_\tau}\big)^+\big]. \end{aligned}$$

Up to the factor $K$ in front of the supremum, this is nothing else than the McKean optimal stopping problem with strike $K^{-1}e^\epsilon$. The following result then follows directly from Corollary 9.3 in [10].

**Proposition 3.1** *Fix $\epsilon \in \mathbb{R}$ and assume that $s \geq \epsilon$. The solution of* (1.3) *is given by*

$$V_\epsilon^*(x,s) = e^\epsilon Z^{(q)}(x - x_\epsilon^*) - Ke^x Z_1^{(q-\psi(1))}(x - x_\epsilon^*),$$

*where*

$$x_\epsilon^* := \epsilon + \begin{cases} \log\big(K^{-1}\frac{q}{\Phi(q)}\frac{\Phi(q)-1}{q-\psi(1)}\big), & q \neq \psi(1), \\ \log\big(K^{-1}\frac{q}{\psi'(1)}\big), & q = \psi(1), \end{cases}$$

*and corresponding optimal stopping time* $\tau_\epsilon^* := \inf\{t \geq 0 : X_t < x_\epsilon^*\}$.

Next, define the quantity

$$\eta := \begin{cases} \log\big(K\frac{\Phi(q)}{q}\frac{q-\psi(1)}{\Phi(q)-1}\big), & q \neq \psi(1), \\ \log\big(K\frac{\psi'(1)}{q}\big), & q = \psi(1), \end{cases} \tag{3.1}$$

and note that $\epsilon - x_\epsilon^* = \eta$. Moreover, Eq. (8.2) in [10] states that

$$\mathbb{E}\big[e^{\underline{X}_{\mathbf{e}_q}}\big] = \begin{cases} \frac{q}{\Phi(q)}\frac{\Phi(q)-1}{q-\psi(1)}, & q \neq \psi(1), \\ \frac{q}{\psi'(1)}, & q = \psi(1), \end{cases} \tag{3.2}$$

where $\underline{X}_{\mathbf{e}_q} = \inf_{0 \leq u \leq \mathbf{e}_q} X_u$ and $\mathbf{e}_q$ is an exponential random variable with parameter $q > 0$ independent of $X$. In particular, the terms on the right-hand side of (3.2) are smaller than or equal to one.

Now we want to investigate the solution of (1.3) for $s < \epsilon$. To this end, assume temporarily that $\epsilon < x_\epsilon^*$ or, equivalently, $\eta < 0$, and hence $K < 1$ which implies that $e^{-qt}(e^{\overline{X}_t \wedge \epsilon} - Ke^{X_t})^+ = e^{-qt}(e^{\overline{X}_t \wedge \epsilon} - Ke^{X_t})$ as long as $\overline{X}_t \leq \epsilon$. We are now going to argue in the same way as described in [17], Sect. 3, page 6: The dynamics of $(X, \overline{X})$ are such that $\overline{X}$ remains constant at times when $X$ is undertaking an excursion away from $\overline{X}$. Although $e^{\overline{X}_t \wedge \epsilon} - Ke^{X_t}$ increases with the depth of the excursion, the payoff during an excursion is bounded above by $e^s$, where $s$ is the current value of $\overline{X}$ during the excursion. Due to the exponential discounting, one should therefore not allow $X$ to drop too far below $\overline{X}$, as otherwise the time it will take $X$ to recover and reach $s$ will prove costly in terms of gain. Hence, given that $\overline{X}$ is at level $s$, there should be a point $g_\epsilon(s) > 0$ such that if the process $X$ reaches or jumps below the value $s - g_\epsilon(s)$, we should stop. In more mathematical terms, we expect, as long as $\overline{X} < \epsilon$, an optimal strategy of the form

$$\inf\big\{t \geq 0 : \overline{X}_t - X_t \geq g_\epsilon\big(\overline{X}_t\big)\big\} \tag{3.3}$$

for some decreasing function $g_\epsilon : (-\infty, \epsilon) \to [0, \infty)$. Once $\overline{X}$ reaches level $\epsilon$, Proposition 3.1 says that one should stop immediately as $\epsilon < x_\epsilon^*$. This means that $g_\epsilon$ has to satisfy the additional requirement $\lim_{s \uparrow \epsilon} g_\epsilon(s) = 0$. Summing up, we expect an optimal stopping time of the form

$$\rho_\epsilon = \inf\big\{t \geq 0 : \big(X_t, \overline{X}_t\big) \notin C_1 \cup C_2\big\}, \tag{3.4}$$

where $C_1 := \{(x, s) \in E : s < \epsilon \text{ and } s - x < g_\epsilon(s)\}$ and $C_2 := \{(x, s) \in E : x > x_\epsilon^*\}$. The set $C_1 \cup C_2$ is usually called continuation region, and it is shown in the drawing on the left-hand side in Fig. 2.

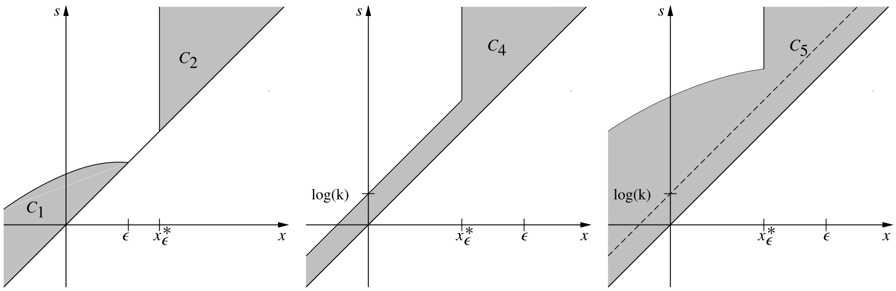


**Fig. 2** *Left*: anticipated continuation and stopping regions when $\epsilon < x_\epsilon^*$. *Middle*: the set $C_4$ which is necessarily contained in the continuation region when $\epsilon \geq x_\epsilon^*$ and $K > 1$. *Right*: anticipated continuation and stopping regions when $\epsilon \geq x_\epsilon^*$ and $K > 1$

Now assume that $\epsilon \geq x_\epsilon^*$ or, equivalently, $\eta \geq 0$, and that $K > 1$. Then the situation looks quite different. Since $K > 1$, we see that $e^{-qt}(e^{\overline{X}_t} - Ke^{X_t})^+ = 0$ whenever $(X, \overline{X})$ lies in the strip $C_3 := \{(x, s) \in E : s - \log K \leq x\}$, and therefore it is never optimal to stop as long as the process $(X, \overline{X})$ lies in $C_3$. Combining this with Proposition 3.1, we see that the continuation region must at least contain the set $C_4 := C_3 \cup \{(x, s) \in E : x > x_\epsilon^*\}$; see the middle drawing in Fig. 2. The whole discussion in the previous paragraph applies here as well, except that one has to take into account the strip $C_3$. In other words, we look again for stopping strategies of the form (3.3) as long as $\overline{X} < \epsilon$, but the boundary condition $\lim_{s\uparrow\epsilon} g_\epsilon(s) = 0$ should be replaced by $\lim_{s\uparrow\epsilon} g_\epsilon(s) = \eta = \epsilon - x_\epsilon^* \geq 0$. The anticipated continuation region

$$C_5 := \{(x, s) \in E \mid s \leq \epsilon \text{ and } s - g_\epsilon(s) < x \text{ or } x > x_\epsilon^*\}$$

is pictorially displayed on the right-hand side in Fig. 2. Finally, if $\epsilon \geq x_\epsilon^*$ and $K \leq 1$, a similar reasoning applies except that there will be no strip $C_3$.

The discussion so far leaves us with two questions:

– How to choose $g_\epsilon$?
– Given $g_\epsilon$, what is $\mathbb{E}_{x,s}[e^{-q\rho_\epsilon}(e^{\overline{X}_{\rho_\epsilon}\wedge\epsilon} - Ke^{X_{\rho_\epsilon}})^+]$, where $\rho_\epsilon$ is either as in (3.4) or $\rho_\epsilon = \inf\{t \geq 0 : (X_t, \overline{X}_t) \notin C_5\}$?

These questions can be answered with the help of the so-called principle of smooth and continuous fit [14, 18, 19] which will provide an ordinary differential equation characterising $g_\epsilon$ and a candidate value function. The details are given in Sect. 6.

## 4 Main results

This section is the verification part of our "guess and verify" approach. Given the candidate solution derived in Sects. 3 and 6, we now verify that it is indeed a solution. The proofs of all the results presented in this section are given in Sect. 7.

We begin by introducing an auxiliary function $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(z) := Z^{(q)}(z) - \Big(q - K\big(q - \psi(1)\big)e^{-z}\Big)W^{(q)}(z).$$

This function will play an important role throughout the remainder of this article and hence we spend some time investigating some of its properties. However, before we can do this, define

$$\beta_0 := \begin{cases} \log(K(q-\psi(1))/q), & \text{if } q > \psi(1), \\ -\infty, & \text{if } q \le \psi(1). \end{cases}$$

In particular, note that the function $y \mapsto q - K(q-\psi(1))e^{-y}$, $y \ge 0$, is strictly negative on $[0, \beta_0 \vee 0)$ and positive on $[\beta_0 \vee 0, \infty)$. Also observe that $\beta_0 \vee 0 \le \eta \vee 0$. This is clear if $q \le \psi(1)$, and if $q > \psi(1)$ we need to show that $\beta_0 \ge 0$ implies $\eta \ge \beta_0$. Indeed, by the definition of $\eta$, we have

$$q - K\big(q-\psi(1)\big)e^{-\eta} = q/\Phi(q) > 0$$

and therefore, by the definition of $\beta_0$, it follows that $\eta > \beta_0$. We can now state our first result concerning $f$.

**Lemma 4.1** *Suppose that $q > \psi(1)$. Then $f$ is strictly increasing on $(0, \beta_0 \vee 0]$ and strictly decreasing on $(\beta_0 \vee 0, \infty)$. Moreover, the function $f$ tends to $-\infty$ as $z \to \infty$.*

Next, denote by $\mathcal{G}$ the general class of spectrally negative Lévy processes and define the subclass

$$\mathcal{H}_{q,K} := \big\{X \in \mathcal{G} : X \text{ is of unbounded variation} \\ \text{or } X \text{ is of bounded variation with } \mathtt{d} > q - K\big(q-\psi(1)\big)\big\}.$$

Furthermore, define the quantity

$$k^* := \inf\{z > \eta \vee 0 : f(z) \le 0\} \in [0, \infty], \tag{4.1}$$

where $\eta$ was defined in (3.1) and we set $\inf \varnothing = \infty$.

**Lemma 4.2**

(a) *If $q > \psi(1)$ and $X \in \mathcal{H}_{q,K}$, then $k^* \in (\eta \vee 0, \infty)$.*
(b) *If $q > \psi(1)$ and $X \in \mathcal{G} \setminus \mathcal{H}_{q,K}$, then $k^* = 0$.*
(c) *If $q \le \psi(1)$, then $k^* = \infty$.*

We are now in a position to define the function $g_\epsilon$ which will, as we shall see in due course, describe the optimal boundary of (1.3).

**Lemma 4.3** *Fix $\epsilon \in \mathbb{R}$. Moreover, suppose that $q > \psi(1)$ and $X \in \mathcal{H}_{q,K}$ or that $q \le \psi(1)$. Then there exists a unique solution $g_\epsilon : (-\infty, \epsilon) \to (\eta \vee 0, k^*)$ of the differential equation*

$$g'_\epsilon(s) = 1 - \frac{Z^{(q)}(g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))(q - K(q-\psi(1))e^{-g_\epsilon(s)})} \quad \text{on } (-\infty, \epsilon) \tag{4.2}$$

*satisfying* $\lim_{s\uparrow\epsilon} g_\epsilon(s) = \eta \vee 0$. *In particular,* $\lim_{s\downarrow-\infty} g_\epsilon(s) = k^*$.

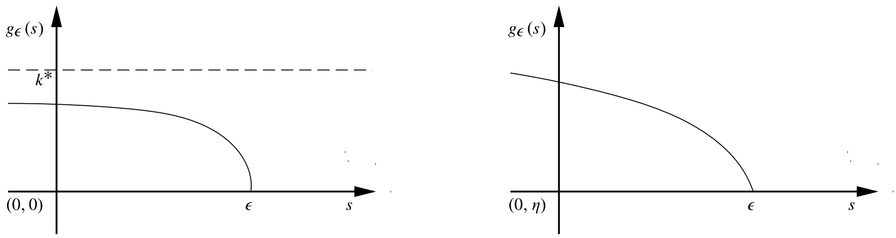


**Fig. 3** In both pictures, it is supposed that $X$ is of unbounded variation. However, on the left-hand side we additionally assume that $q > \psi(1)$ [and hence $k^* \in (\eta \vee 0, \infty)$] and $\eta < 0$, whereas on the right-hand side it is assumed that $q \le \psi(1)$ [and hence $k^* = \infty$] and $\eta > 0$

*On the other hand, when $q > \psi(1)$ and $X \in \mathcal{G} \setminus \mathcal{H}_{q,K}$, we adopt the convention that $g_\epsilon(s) = k^* = 0$ for $s \in (-\infty, \epsilon)$.*

It is possible to say a bit more about the function $g_\epsilon$ in the case when $q > \psi(1)$ and $X \in \mathcal{H}_{q,K}$, or when $q \le \psi(1)$. Specifically, with the help of (2.4) and Lemma 3.3 in [9], one obtains

$$\lim_{s\uparrow\epsilon} g_\epsilon'(s) = 1 - \frac{Z^{(q)}(\eta \vee 0)}{W^{(q)}(\eta \vee 0)(q - K(q-\psi(1))e^{-(\eta\vee 0)})}$$

and

$$\lim_{s\downarrow-\infty} g_\epsilon'(s) = \begin{cases} 0, & \text{if } q > \psi(1) \text{ and } X \in \mathcal{H}_{q,K}, \\ 1 - \Phi(q)^{-1}, & \text{if } q \le \psi(1). \end{cases}$$

Note in particular that $\lim_{s\uparrow\epsilon} g_\epsilon'(s) = -\infty$ whenever $\eta \le 0$ and $X$ is of unbounded variation and that this cannot happen when $X$ is of bounded variation because we have $W^{(q)}(0) = \mathtt{d}^{-1} > 0$. Put differently, the shape of $g_\epsilon$ at $\epsilon$ may change according to the path variation of $X$. A similar observation has already been made in [15] which treats (1.3) for $K = 0$. The differences in the behaviour of $g_\epsilon$ are illustrated in Fig. 3.

In order to state the main result, we need some more notation. Define the continuation regions

$$\begin{aligned} C_I^* = C_{I,g_\epsilon}^* &:= \{(x,s) \in E \mid s \le \epsilon \text{ and } s - g_\epsilon(s) < x \le s\}, \\ C_{II}^* = C_{II,\epsilon}^* &:= \{(x,s) \in E \mid x > x_\epsilon^*\} \end{aligned}$$

and the stopping region $D^* = D_{g_\epsilon}^* = E \setminus (C_I^* \cup C_{II}^*)$. Note that if $q > \psi(1)$ and $X \in \mathcal{G} \setminus \mathcal{H}_{q,K}$, then $C_I^* = \varnothing$.

**Theorem 4.4** *Fix $\epsilon \in \mathbb{R}$. The solution of (1.3) is given by*

$$V_\epsilon^*(x,s) = \begin{cases} e^\epsilon Z^{(q)}(x - x_\epsilon^*) - K e^x Z_1^{(q-\psi(1))}(x - x_\epsilon^*), & s \ge \epsilon, \\ e^s Z^{(q)}(x - s + g_\epsilon(s)) - K e^x Z_1^{(q-\psi(1))}(x - s + g_\epsilon(s)), & s < \epsilon, \end{cases}$$

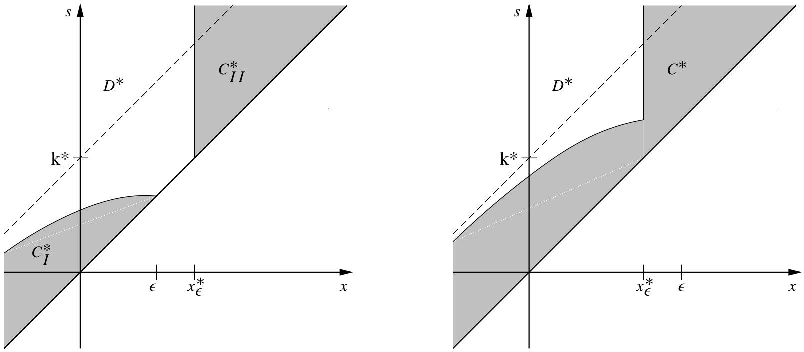


**Fig. 4** In both pictures, it is supposed that $X$ is of unbounded variation and $q > \psi(1)$. The difference is that on the left-hand side, we have $\epsilon < x^*_\epsilon$ which leads to a continuation region consisting of two components, whereas on the right-hand side, we have $\epsilon > x^*_\epsilon$ resulting in a connected continuation region

*with corresponding optimal strategy* $\rho^*_\epsilon := \inf\{t \geq 0 : (X_t, \overline{X}_t) \in D^*_{g_\epsilon}\}$ *and* $g_\epsilon$ *as in Lemma* 4.3.

Some examples for the stopping and continuation regions are pictorially displayed in Fig. 4. In particular, let us emphasise that the continuation region is connected if and only if $\epsilon > x^*_\epsilon$ or, equivalently, $\eta > 0$; otherwise it consists of two disjoint components. Moreover, in the case when $\epsilon > x^*_\epsilon$, one sees that the process $(X, \overline{X})$ has to squeeze through a "bottleneck" to get into the region where the second component of $(X, \overline{X})$ is larger than or equal to $\epsilon$. It is this "special" feature of the continuation region that has motivated the name "bottleneck option" for payoffs of type (1.1). Also note that provided $X \in \mathcal{H}_{q,K}$, it follows from the definition of $\eta$ in (3.1) that the critical value in order to see a bottleneck or not is given by $K = \frac{q(\Phi(q)-1)}{\Phi(q)(q-\psi(1))}$ if $q \neq \psi(1)$ and $K = \frac{q}{\psi'(1)}$ if $q = \psi(1)$.

It is also interesting to investigate what happens if no cap is present, that is, if $\epsilon = \infty$. In this case, problem (1.3) reads

$$V^*_\infty(x, s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}\left[e^{-q\tau}\left(e^{\overline{X}_\tau} - Ke^{X_\tau}\right)^+\right]. \tag{4.3}$$

By a change of measure as in (2.2), one could now reduce this problem to a one-dimensional optimal stopping problem for the reflected process $Y = \{Y_t : t \geq 0\}$, where $Y_t = \overline{X}_t - X_t$ [see [2] for a very similar argument in the case when $K = 0$ in (4.3)]. In this case, the general theory of optimal stopping [19] suggests that the optimal stopping time is an upcrossing time of the process $Y$ at a certain constant level. This is indeed the case, and one could in principle prove this by actually solving the resulting one-dimensional optimal stopping problem for $Y$. Here, however, we solve (4.3) with the help of the work already done in Theorem 4.4 and a simple limiting procedure.

**Corollary 4.5** *Assume that* $\epsilon = \infty$.

(i) *Suppose that* $q > \psi(1)$. *The solution of* (1.3) *is given by*

$$V_\infty^*(x,s) = e^s Z^{(q)}(x - s + k^*) - K e^x Z_1^{(q-\psi(1))}(x - s + k^*)$$

*with corresponding optimal strategy* $\rho_\infty^* := \inf\{t \geq 0 : \overline{X}_t - X_t \geq k^*\}$, *where* $k^* \in [0,\infty)$ *is defined in* (4.1).

(ii) *If* $q \leq \psi(1)$, *then there is no solution to* (1.3), *and* $V_\infty^*(x,s) \equiv \infty$.

Observe that if $q \leq \psi(1)$, then the value function is equal to infinity. Of course, this is not possible in the presence of a cap $\epsilon \in \mathbb{R}$.

*Remark 4.6* In [15], the problem (1.3) is studied for $K = 0$ and one should, at least formally, be able to recover those results by letting $K$ tend to zero in Theorem 4.4 and Corollary 4.4. This is indeed the case and follows in a straightforward way from the fact that $x_\epsilon^* \to \infty$ and $\eta \to -\infty$ as $K \to 0$.

## 5 Example

The solution of (1.3) in Theorems 4.4 and 4.5 is given semi-explicitly in terms of scale functions and a specific solution $g_\epsilon$ of the ordinary differential equation (4.2). A first step towards more explicit solutions of (1.3) is to look at processes $X$ where explicit expressions for $W^{(q)}$ and $Z^{(q)}$ are available. In recent years, various authors have found several processes whose scale functions are explicitly known; for instance, see Example 1.3 as well as Chaps. 4 and 5 in [9]. Here we consider one example where $X$ has jumps. Specifically, suppose that $X$ is an $\alpha$-stable process with Laplace exponent $\psi(\theta) = \theta^\alpha, \theta \geq 0$, where $\alpha \in (1,2]$. Moreover, suppose that $q > \psi(1)$ which in this case means that $q > 1$. It is known from Example 4.17 of [9] and Sect. 8.3 of [2] that for $x \geq 0$,

$$W^{(q)}(x) = x^{\alpha-1} E_{\alpha,\alpha}(qx^\alpha) \quad \text{and} \quad Z^{(q)}(x) = E_{\alpha,1}(qx^\alpha),$$

where $E_{\alpha,\beta}$ is the two-parameter Mittag–Leffler function defined for $\alpha > 0$, $\beta > 0$ as

$$E_{\alpha,\beta}(x) = \sum_{n=0}^\infty \frac{x^n}{\Gamma(\alpha n + \beta)}.$$

By the definition of $Z_1^{(q)}$ [see (2.6)] and (2.3), we obtain

$$Z_1^{(q-\psi(1))}(x) = 1 + \big(q - \psi(1)\big) \int_0^x e^{-y} W^{(q)}(y)\,dy, \quad x \geq 0.$$

In order to compute the stopping boundary, one might try to solve (4.2) numerically, but this is not straightforward as there is no initial point to start a numerical scheme from. Moreover, the possibility of $g_\epsilon$ having infinite gradient at $\epsilon$ might lead to inaccuracies in the numerical scheme. Therefore, we follow a different route, which avoids these difficulties. Instead of looking at $g_\epsilon$, we rather focus on its inverse (see

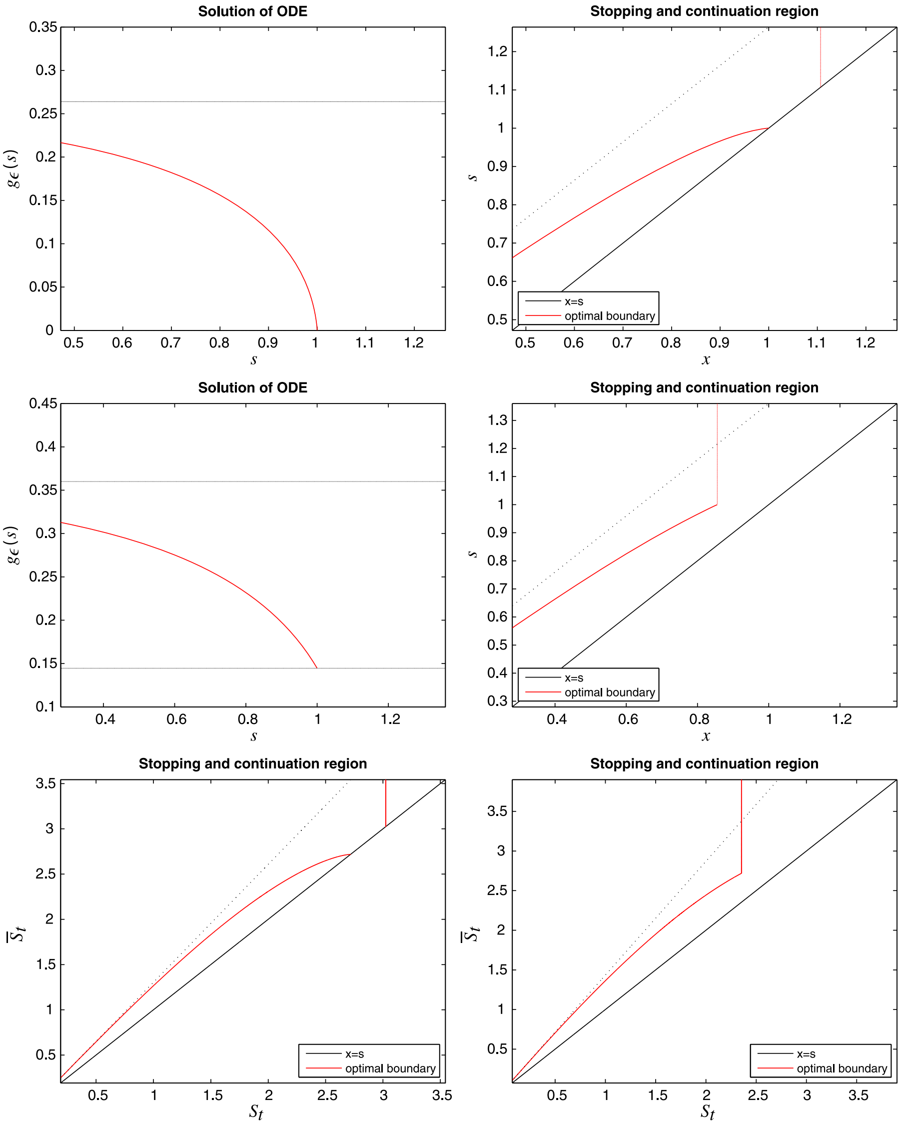


**Fig. 5** *Top two pictures*: A visualisation of $s \mapsto g_\epsilon(s)$ and the resulting optimal boundary when $q = 3$, $\epsilon = 1$, $K = 0.7$ and $\alpha = 1.5$. It follows that $x_\epsilon^* \approx 1.11$, $\eta \approx -0.11$ and $k^* \approx 0.26$. *Middle two pictures*: A visualisation of $s \mapsto g_\epsilon(s)$ and the resulting optimal boundary when $q = 3$, $\epsilon = 1$, $K = 0.9$ and $\alpha = 1.5$. It follows that $x_\epsilon^* \approx 0.86$, $\eta \approx 0.14$ and $k^* \approx 0.36$. *Bottom two pictures*: The corresponding continuation and stopping regions for the original problem for $(S, \overline{S})$ with $C = e$

the proof of Lemma 4.3 in Sect. 7 below)

$$H(s) = \epsilon - \int_{\eta \vee 0}^{s} \frac{(q - K(q-\psi(1))e^{-u})W^{(q)}(u)}{Z^{(q)}(u) - (q - K(q-\psi(1))e^{-u})W^{(q)}(u)}\, du, \quad s \in (\eta \vee 0, k^*), \tag{5.1}$$

where $k^* \in (0, \infty)$ is the unique root of

$$Z^{(q)}(z) - \Big(q - K\big(q - \psi(1)\big)e^{-z}\Big)W^{(q)}(z) = 0.$$

Since $H$ is the inverse of $g_\epsilon$, plotting $(H(y), y)$, $y \in (\eta \vee 0, k^*)$, yields visualisations of $s \mapsto g_\epsilon(s)$ for $s \in (-\infty, \epsilon)$; see Fig. 5. Similarly, plotting $(H(y) - y, H(y))$, $y \in (\eta \vee 0, k^*)$, produces visualisations of the optimal stopping boundary in the $(x, s)$-plane; see Fig. 5. Further, to obtain the continuation and stopping regions for the original problem involving $S$ and $\overline{S}$, one only needs to plot $(\exp(H(y) - y),$ $\exp(H(y)))$ for $y \in (\eta \vee 0, k^*)$; see Fig. 5. Because we are unable to compute the integral in (5.1) explicitly, we use numerical integration in Matlab to obtain an approximation of the integral. We also use Matlab to compute the Mittag–Leffler function (cf. [20]) and to solve the equation for $k^*$.

Of course, once one starts to compute things numerically, there are many more examples that could be looked at—for instance the Black–Scholes case where $X$ corresponds to a linear Brownian motion, or the case where $X$ is a jump-diffusion. Similar results in this direction for a slightly different problem have been obtained in [15] and could be carried over to the setting here in a straightforward way.

## 6 Guess via principle of smooth and continuous fit

The goal of this section is to answer the two questions raised at the end of Sect. 3. The argument presented here is an adaptation of [17] to our setting. It has already been successfully applied in [11, 15] in similar or related situations. The difference from [11, 15], however, is that here the payoff also depends on $X$ and not only $\overline{X}$. As we shall see in due course, this can be dealt with by a change of measure which essentially puts one back into the situation where the payoff only depends on $\overline{X}$. **Throughout this section, we assume that $s < \epsilon$. Moreover, for simplicity, suppose that $q > \psi(1)$.**

To begin with, assume that $X$ is of unbounded variation. We deal with the bounded variation case later. From the general theory of optimal stopping (cf. Sect. 13 of [19]), we informally expect the value function

$$U_\epsilon(x, s) := \mathbb{E}_{x,s}\big[e^{-q\rho_\epsilon}\big(e^{\overline{X}_{\rho_\epsilon} \wedge \epsilon} - Ke^{X_{\rho_\epsilon}}\big)^+\big],$$

where $\rho_\epsilon$ was defined in Sect. 3, to satisfy the system

$$\begin{aligned} \varGamma U_\epsilon(x, s) &= qU_\epsilon(x, s) \quad \text{for } s - g_\epsilon(s) < x < s \text{ with } s \text{ fixed}, \\ \frac{\partial U_\epsilon}{\partial s}(x, s)\bigg|_{x=s-} &= 0 \qquad\qquad\qquad \text{(normal reflection)}, \\ U_\epsilon(x, s)\big|_{x=(s-g_\epsilon(s))+} &= e^s - Ke^{s-g_\epsilon(s)} \qquad \text{(instantaneous stopping)}, \end{aligned} \tag{6.1}$$

where $\varGamma$ is the infinitesimal generator of the process $X$ under $\mathbb{P}_0$. Moreover, the principle of smooth fit [14, 19] suggests that this system should be complemented by

$$\lim_{x \downarrow s - g_\epsilon(s)} \frac{\partial U_\epsilon}{\partial x}(x, s) = -Ke^{s-g_\epsilon(s)} \quad \text{(smooth fit)}. \tag{6.2}$$

Note that although the smooth fit condition is not necessarily part of the general theory, it is imposed since by the "rule of thumb" outlined in Sect. 7 in [1], it should hold in this setting because of path regularity. This belief will be vindicated when we show that the system (6.1) together with (6.2) leads to the solution of (1.3).

Next, splitting over the events $\{\rho_\epsilon < \tau_s^+\}$ and $\{\rho_\epsilon > \tau_s^+\}$ in the first equality and applying the strong Markov property at $\tau_s^+$ and a change of measure according to (2.2) in the second equality gives

$$\begin{aligned}
U_\epsilon(x,s) &= e^s \mathbb{E}_{x,s}\Big[e^{-q\tau^-_{s-g_\epsilon(s)}} 1_{\{\tau^-_{s-g_\epsilon(s)}<\tau_s^+\}}\Big] \\
&\quad - K\mathbb{E}_{x,s}\Big[e^{-q\tau^-_{s-g_\epsilon(s)}+X_{\tau^-_{s-g(s)}}} 1_{\{\tau^-_{s-g_\epsilon(s)}<\tau_s^+\}}\Big] \\
&\quad + \mathbb{E}_{x,s}\Big[e^{-q\rho_\epsilon}\big(e^{\overline{X}_{\rho_\epsilon}\wedge\epsilon} - Ke^{X_{\rho_\epsilon}}\big)^+ 1_{\{\tau^-_{s-g(s)}>\tau_s^+\}}\Big] \\
&= e^s \mathbb{E}_{x,s}\Big[e^{-q\tau^-_{s-g_\epsilon(s)}} 1_{\{\tau^-_{s-g_\epsilon(s)}<\tau_s^+\}}\Big] \\
&\quad - Ke^x \mathbb{E}^1_{x,s}\Big[e^{-(q-\psi(1))\tau^-_{s-g_\epsilon(s)}} 1_{\{\tau^-_{s-g_\epsilon(s)}<\tau_s^+\}}\Big] \\
&\quad + \mathbb{E}_{x,s}\Big[e^{-q\tau_s^+} 1_{\{\tau^-_{s-g_\epsilon(s)}>\tau_s^+\}}\Big] U_\epsilon(s,s).
\end{aligned}$$

Furthermore, using Proposition 1 of [2] and rearranging terms in the first equality and applying (2.3) in the second equality shows that

$$\begin{aligned}
U_\epsilon(x,s) &= e^s Z^{(q)}\big(x-s+g_\epsilon(s)\big) - Ke^x Z_1^{(q-\psi(1))}\big(x-s+g_\epsilon(s)\big) \\
&\quad - e^s W^q\big(x-s+g_\epsilon(s)\big)\frac{Z^{(q)}(g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))} \\
&\quad + \frac{W^{(q)}(x-s+g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))} U_\epsilon(s,s) \\
&\quad + Ke^x W_1^{(q-\psi(1))}\big(x-s+g_\epsilon(s)\big)\frac{Z_1^{(q-\psi(1))}(g_\epsilon(s))}{W_1^{(q-\psi(1))}(g_\epsilon(s))} \\
&= e^s Z^{(q)}\big(x-s+g_\epsilon(s)\big) - Ke^x Z_1^{(q-\psi(1))}\big(x-s+g_\epsilon(s)\big) \\
&\quad - e^s W^{(q)}\big(x-s+g_\epsilon(s)\big)\frac{Z^{(q)}(g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))} \\
&\quad + \frac{W^{(q)}(x-s+g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))} U_\epsilon(s,s) \\
&\quad + Ke^s W^{(q)}\big(x-s+g_\epsilon(s)\big)\frac{Z_1^{(q-\psi(1))}(g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))}.
\end{aligned}$$

The smooth fit condition in (6.2) now implies that

$$\frac{W^{(q)\prime}(x-s+g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))}\Big(e^s Z^{(q)}\big(g_\epsilon(s)\big)-U_\epsilon(s,s)-Ke^s Z_1^{(q-\psi(1))}\big(g_\epsilon(s)\big)\Big)\longrightarrow 0$$

as $x\downarrow s-g_\epsilon(s)$. However, by (2.5) the first factor tends to a strictly positive value or infinity which shows that

$$U_\epsilon(s,s)=e^s Z^{(q)}\big(g_\epsilon(s)\big)-Ke^s Z_1(q-\psi(1))\big(g_\epsilon(s)\big).$$

This would mean that for $(x,s)\in E$ such that $s-g_\epsilon(s)<x<s$, we have

$$U_\epsilon(x,s)=e^s Z^{(q)}\big(x-s+g_\epsilon(s)\big)-Ke^x Z_1^{(q-\psi(1))}\big(x-s+g_\epsilon(s)\big).$$

Having derived the form of a candidate optimal value function $U_\epsilon$, we still need to do the same for $g_\epsilon$. Using the normal reflection condition (6.1) shows that our candidate function $g_\epsilon$ should satisfy the differential equation

$$g_\epsilon'(s)=1-\frac{Z^{(q)}(g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))(q-K(q-\psi(1))e^{-g_\epsilon(s)})}\quad\text{on }(-\infty,\epsilon).$$

If $X$ is of bounded variation, we informally expect from the general theory that $U_\epsilon$ satisfies the first two equations of (6.1). Additionally, the principle of continuous fit [1, 18] suggests that the system should be complemented by

$$\lim_{x\downarrow s-g_\epsilon(s)}U_\epsilon(x,s)=e^s-Ke^{s-g_\epsilon(s)}\quad\text{(continuous fit).}$$

A very similar argument as above produces the same candidate value function and the same ordinary differential equation for $g_\epsilon$.

It remains to check that the heuristic argument presented above leads to the solution of (1.3)—this is essentially the content of Theorem 4.4.

## 7 Proofs

*Proof of Lemma 4.1* Using the assumed regularity of $W^{(q)}$ and the relation (2.3) in the second equality, one sees that

$$\begin{aligned}f'(z)&=\Big(q-K\big(q-\psi(1)\big)e^{-z}\Big)\big(W^{(q)}(z)-W^{(q)\prime}(z)\big)\\&=\Big(q-K\big(q-\psi(1)\big)e^{-z}\Big)e^{\Phi(q)z}\Big(W_{\Phi(q)}(z)\big(1-\Phi(q)\big)-W_{\Phi(q)}'(z)\Big).\end{aligned}$$

Since $\Phi(q)>1$, we have $W_{\Phi(q)}(z)(1-\Phi(q))-W_{\Phi(q)}'(z)<0$ for $z>0$ and hence the stated monotonicity properties of $f$ follow from the properties of the map $z\mapsto q-K(q-\psi(1))e^{-z}$ stated just before Lemma 4.1. As for the behaviour of $f(z)$

for large $z$, we infer from Lemma 3.3 in [9] that

$$\lim_{z\to\infty} f(z)/\big(qW^{(q)}(z)\big) = \Phi(q)^{-1} - 1.$$

Again using (2.3), we have $W^{(q)}(z) = e^{\Phi(q)z}W_{\Phi(q)}(z)$ which tends to infinity as $z \to \infty$. As $\Phi(q) > 1$, we conclude that $\lim_{z\to\infty} f(z) = -\infty$. □

*Proof of Lemma 4.2* (a) First suppose that $X$ has paths of unbounded variation. By (2.4) this necessarily means that $W^{(q)}(0+) = 0$. Thus we see that $f(0+) = 1$ and the existence of a unique root $k^* > 0$ of $f(z) = 0$ is guaranteed by Lemma 4.1 and the intermediate value theorem. Moreover, one needs to check whether $k^* > \eta$ whenever $\eta > 0$. Since $k^*$ is a root of $f(z) = 0$, we have

$$\frac{Z^{(q)}(k^*)}{W^{(q)}(k^*)} = q - K\big(q - \psi(1)\big)e^{-k^*}. \tag{7.1}$$

Since the map $z \mapsto Z^{(q)}(z)/W^{(q)}(z)$, $z > 0$, is decreasing (cf. Eq. (45) of [9]) and because of Lemma 3.3 in [9], the left-hand side of (7.1) is (strictly) bounded below by $q/\Phi(q)$. Hence, after some algebra, one sees that

$$k^* > \log\left(K\frac{\Phi(q)}{q}\frac{q-\psi(1)}{\Phi(q)-1}\right) = \eta.$$

Now suppose that $X$ has paths of finite variation and $\mathtt{d} > q - K(q - \psi(1))$. In this case we see that $f(0+) > 0$. Using Lemma 4.1 in conjunction with the intermediate value theorem shows again that there exists a unique root $k^* > 0$ of $f(z) = 0$. The fact that $k^* > \eta$ whenever $\eta > 0$ follows as above.

(b) The fact that $0 < \mathtt{d} \le q - K(q - \psi(1))$ implies on the one hand that $f(0+) \le 0$ and on the other hand that $K < q/(q - \psi(1))$. By Lemma 4.1, we therefore have $f(z) < 0$ for $z > 0$. To conclude that $k^* = 0$, it remains to check that $\eta \le 0$. Since $\mathtt{d} \le q - K(q - \psi(1))$, we have $K \le (q - \mathtt{d})/(q - \psi(1))$ and hence

$$\eta = \log\left(K\frac{\Phi(q)}{q}\frac{q-\psi(1)}{\Phi(q)-1}\right) \le \log\left(\frac{\Phi(q)}{q}\frac{q-\mathtt{d}}{\Phi(q)-1}\right).$$

It follows that $\eta \le 0$ provided that $\frac{\Phi(q)}{q}\frac{q-\mathtt{d}}{\Phi(q)-1} \le 1$ or, equivalently, $q/\Phi(q) \le \mathtt{d}$. Indeed, since $\Phi(q) > 0$ one sees with the help of (2.1) that

$$\frac{q}{\Phi(q)} = \frac{\psi(\Phi(q))}{\Phi(q)} = \mathtt{d} - \frac{1}{\Phi(q)}\int_{(-\infty,0)}\big(1 - e^{\Phi(q)x}\big)\,\Pi(dx) \le \mathtt{d}.$$

(c) First assume that $q < \psi(1)$ and assume for a contradiction that there exists a $z_0 > \eta \vee 0$ such that $f(z_0) \le 0$. Since $Z^{(q)}(z_0)/W^{(q)}(z_0)$ is bounded below by $q/\Phi(q)$ [as explained in (a)], it follows that

$$\frac{q}{\Phi(q)} < q - K\big(q - \psi(1)\big)e^{-z_0}$$

or, after some straightforward algebra and using that $q < \psi(1)$,

$$z_0 < \log\left(K \frac{\Phi(q)}{q} \frac{q-\psi(1)}{\Phi(q)-1}\right) = \eta.$$

This is a contraction to $z_0 \geq \eta \vee 0$, and hence $f(z) > 0$ for $z > \eta \vee 0$. In other words, $k^* = \infty$. Finally, if $q = \psi(1)$, we have $f(z) = Z^{(q)}(z) - qW^{(q)}(z) > 0$ for $z > 0$ by Eq. (42) of [9] and hence again $k^* = \infty$. □

*Proof of Lemma 4.3* The proof is very similar to the proof of Lemma 4.1 in [15]. The idea is to construct the solution $g_\epsilon$ by defining a suitable bijection from $(\eta \vee 0, k^*)$ to $(-\infty, \epsilon)$ whose inverse satisfies the differential equation and the boundary conditions. We present the case when $q > \psi(1)$ and $X \in \mathcal{H}_{q,K}$. The case when $q \leq \psi(1)$ follows analogously to the proof of Lemma 4.1 in [15].

Assume that $q > \psi(1)$ and $X \in \mathcal{H}_{q,K}$. The fact that $\eta \vee 0 \geq \beta_0 \vee 0$ (see the discussion just before Lemma 4.1) and Lemmas 4.1 and 4.2 imply that $k^* \in (\eta \vee 0, \infty)$ and that the function

$$s \mapsto h(s) := 1 - \frac{Z^{(q)}(s)}{W^{(q)}(s)(q - K(q-\psi(1))e^{-s})}$$

is strictly negative on $(\eta \vee 0, k^*)$. Moreover, we have $\lim_{s\downarrow\eta\vee 0} h(s) \in [-\infty, 0)$ and $\lim_{s\uparrow k^*} h(s) = 0$. Due to these properties, the function $H : (\eta \vee 0, k^*) \to (-\infty, \epsilon)$ defined by

$$H(s) := \epsilon + \int_{\eta\vee 0}^{s} \frac{1}{h(u)}\,du = \epsilon - \int_{\eta\vee 0}^{s} \frac{W^{(q)}(u)(q - K(q-\psi(1))e^{-u})}{f(u)}\,du$$

is strictly decreasing. If we show that the integral tends to $\infty$ as $s$ approaches $k^*$, we can deduce that $H$ is a bijection from $(\eta \vee 0, k^*)$ to $(-\infty, \epsilon)$. Indeed, by l'Hôspital's rule and due to the fact that $f'(k^*) < 0$, we have

$$\lim_{s\uparrow k^*} \frac{k^*-s}{f(s)} = \frac{-1}{f'(k^*)} =: c > 0.$$

Hence there exist $\delta > 0$ and $s_0 > \eta \vee 0$ such that $c - \delta > 0$ and

$$\frac{1}{f(s)} > \frac{c-\delta}{k^*-s} \quad \text{for } s_0 < s < k^*.$$

Combining this with the fact that

$$\frac{W^{(q)}(u)(q - K(q-\psi(1))e^{-u})}{k^*-u} \geq \frac{W^{(q)}(s_0)(q - K(q-\psi(1))e^{-u})}{k^*-u} \quad \text{for } u > s_0$$

implies that

$$\lim_{s\uparrow k^*} H(s) \leq \epsilon - (c-\delta) \lim_{s\uparrow k^*} \int_{s_0}^{s} \frac{W^{(q)}(u)(q - K(q-\psi(1))e^{-u})}{k^*-u}\,du = -\infty.$$

The discussion above permits us to define $g_\epsilon := H^{-1}$, which lies in the class $C^1((-\infty,\epsilon);(\eta\vee 0,k^*))$. In particular, differentiating $g_\epsilon$ gives

$$g_\epsilon'(s) = \frac{1}{H'(g_\epsilon(s))} = 1 - \frac{Z^{(q)}(g_\epsilon(s))}{qW^{(q)}(g_\epsilon(s))(q-K(q-\psi(1))e^{-g_\epsilon(s)})}$$

for $s\in(-\infty,\epsilon)$, and $g_\epsilon$ satisfies $\lim_{s\to-\infty} g_\epsilon(s)=k^*$ and $\lim_{s\uparrow\epsilon} g_\epsilon(s)=\eta\vee 0$ by construction. Finally, uniqueness follows as in the last part of the proof of Lemma 4.1 in [15]. □

*Proof of Theorem 4.4* Define for $(x,s)\in E$ the function

$$V_\epsilon(x,s) := \begin{cases} e^\epsilon Z^{(q)}(x-x_\epsilon^*) - Ke^x Z_1^{(q-\psi(1))}(x-x_\epsilon^*), & s\geq\epsilon,\\ e^s Z^{(q)}(x-s+g_\epsilon(s)) - Ke^x Z_1^{(q-\psi(1))}(x-s+g_\epsilon(s)), & s<\epsilon. \end{cases}$$

Because of the infinite horizon and Markovian claim structure of (1.3), it is enough to establish the following three results whose proofs are given below:

**Lemma 7.1** *We have $V_\epsilon(x,s)\geq(e^{s\wedge\epsilon}-Ke^x)^+$ for all $(x,s)\in E$.*

**Lemma 7.2** *The process $e^{-qt}V_\epsilon(X_t,\overline{X}_t)$, $t\geq 0$, is a right-continuous $\mathbb{P}_{x,s}$-supermartingale for $(x,s)\in E$.*

**Lemma 7.3** *For all $(x,s)\in E$, we have*

$$V_\epsilon(x,s) = \mathbb{E}_{x,s}\big[e^{-q\rho_\epsilon^*}\big(e^{\overline{X}_{\rho_\epsilon^*}\wedge\epsilon} - Ke^{X_{\rho_\epsilon^*}}\big)^+\big].$$

To see why these three results suffice, note that Lemmas 7.1 and 7.2 together with Fatou's lemma in the second inequality and Doob's stopping theorem in the third inequality show that for $\tau\in\mathcal{M}$ and $(x,s)\in E$,

$$\begin{aligned}\mathbb{E}_{x,s}\big[e^{-q\tau}\big(e^{\overline{X}_\tau\wedge\epsilon}-Ke^{X_\tau}\big)^+\big] &\leq \mathbb{E}_{x,s}\big[e^{-q\tau}V_\epsilon(X_\tau,\overline{X}_\tau)\big]\\ &\leq \liminf_{t\to\infty}\mathbb{E}_{x,s}\big[e^{-q(t\wedge\tau)}V_\epsilon(X_{t\wedge\tau},\overline{X}_{t\wedge\tau})\big]\\ &\leq V_\epsilon(x,s).\end{aligned}$$

In view of Lemma 7.3, this implies $V_\epsilon^*=V_\epsilon$ and that $\rho_\epsilon^*$ is optimal. □

*Proof of Lemma 7.1* Choosing $\tau=0$ in Proposition 3.1 shows that

$$V_\epsilon(x,s)\geq(e^\epsilon-Ke^x)^+$$

for $(x,s)\in E$ such that $s\geq\epsilon$. Hence, we can restrict ourselves to proving the assertion for $x\leq s<\epsilon$.

In a first step, we claim that

$$g_\epsilon(s) \geq \eta \vee 0 \geq \log K \vee 0, \qquad s \in (-\infty, \epsilon). \tag{7.2}$$

If $q > \psi(1)$ and $X \in \mathcal{H}_{q,K}$, or if $q \leq \psi(1)$, then the first inequality in (7.2) holds by construction of $g_\epsilon$ (see Lemma 4.3). On the other hand, if $q > \psi(1)$ and $X \in \mathcal{G} \setminus \mathcal{H}_{q,K}$, we need to show that $\eta \leq 0$ for the first inequality to be true, and this was done in the proof of part (b) of Lemma 4.2. The second inequality follows by the definition of $\eta$ and (3.2).

Next, using (2.3) in the first equality and a change of variables in the second equality, we may rewrite $V_\epsilon(x, s)$ as

$$\begin{aligned} &e^s Z^{(q)}\big(x - s + g_\epsilon(s)\big) - K e^x Z_1^{(q-\psi(1))}\big(x - s + g_\epsilon(s)\big) \\ &= e^s - K e^x + q e^s \int_0^{x-s+g_\epsilon(s)} W^{(q)}(y)\, dy \\ &\quad - K e^x \big(q - \psi(1)\big) \int_0^{x-s+g_\epsilon(s)} e^{-y} W^{(q)}(y)\, dy \\ &= e^s - K e^x + e^s \int_{s-x}^{g_\epsilon(s)} W^{(q)}(y + x - s)\Big(q - K\big(q - \psi(1)\big)e^{-y}\Big)\, dy, \end{aligned} \tag{7.3}$$

where we understand the integral on the right-hand side to be not present whenever $s - x \geq g_\epsilon(x)$. In order to prove the assertion, recall from the discussion just before Lemma 4.1 that

$$\beta_0 \vee 0 \leq \eta \vee 0. \tag{7.4}$$

We now prove the statement of the lemma that the right-hand side of (7.3) is greater than or equal to $(e^s - K e^x)^+$. If $\eta \vee 0 \leq s - x$, then $s - x \geq \log K \vee 0$ by (7.2). Together with (7.4), this implies $V_\epsilon(x, s) \geq e^s - K e^x = (e^s - K e^x)^+$. On the other hand, if $0 \leq s - x < \eta$ (whenever $\eta > 0$), the situation is slightly more complicated as the integrand on the right-hand side of (7.3) might change sign (if $0 < \beta_0 < \eta$) and it is not clear how much the negative and positive parts contribute. To resolve this difficulty, we reduce the problem to an estimate obtained from Proposition 3.1. Specifically, it follows from Proposition 3.1 that

$$\begin{aligned} V_\epsilon^*(\hat{x}, \epsilon) &= e^\epsilon Z^{(q)}(\hat{x} - x_\epsilon^*) - K e^{\hat{x}} Z_1^{(q-\psi(1))}(\hat{x} - x_\epsilon^*) \\ &= e^\epsilon - K e^{\hat{x}} + e^\epsilon \int_0^{\hat{x}-x_\epsilon^*} W^{(q)}(y)\Big(q - K\big(q - \psi(1)\big)e^{\hat{x}-\epsilon-y}\Big)\, dy \\ &\geq (e^\epsilon - K e^{\hat{x}})^+ \end{aligned} \tag{7.5}$$

for $x_\epsilon^* \leq \hat{x} \leq \epsilon$. Now define $\delta := \epsilon - s$ and $\tilde{x} := x + \delta$. In particular, note that $0 \leq s - x < \eta = \epsilon - x_\epsilon^*$ implies $x_\epsilon^* < \tilde{x} \leq \epsilon$. Then, using that $g_\epsilon(s) \geq \eta \geq \beta_0 \vee 0$ in the first and (7.5) with $\hat{x} = \tilde{x}$ in the second inequality, we obtain

$$
\begin{aligned}
&V_\epsilon(x,s)\\
&= e^s - Ke^x + e^s \int_0^{x-s+g_\epsilon(s)} W^{(q)}(y)\Big(q - K\big(q-\psi(1)\big)e^{x-s-y}\Big)\,dy\\
&= e^{-\delta}\bigg(e^\epsilon - Ke^{\tilde{x}} + e^\epsilon \int_0^{\tilde{x}-\epsilon+g_\epsilon(s)} W^{(q)}(y)\Big(q - K\big(q-\psi(1)\big)e^{\tilde{x}-\epsilon-y}\Big)\,dy\bigg)\\
&\geq e^{-\delta}\bigg(e^\epsilon - Ke^{\tilde{x}} + e^\epsilon \int_0^{\tilde{x}-\epsilon+\eta} W^{(q)}(y)\Big(q - K\big(q-\psi(1)\big)e^{\tilde{x}-\epsilon-y}\Big)\,dy\bigg)\\
&= e^{-\delta}\bigg(e^\epsilon - Ke^{\tilde{x}} + e^\epsilon \int_0^{\tilde{x}-x_\epsilon^*} W^{(q)}(y)\Big(q - K\big(q-\psi(1)\big)e^{\tilde{x}-\epsilon-y}\Big)\,dy\bigg)\\
&\geq e^{-\delta}(e^\epsilon - Ke^{\tilde{x}})^+ = (e^s - Ke^x)^+.
\end{aligned}
$$

This completes the proof. □

*Proof of Lemma 7.2* We only prove the result in detail when $X$ has paths of unbounded variation. If it has paths of bounded variation, the proof is similar and we restrict ourselves to only pointing out major changes.

*Unbounded variation case:* As a first step we prove that

$$
e^{-q(t\wedge\tau_\epsilon^+)} V_\epsilon(X_{t\wedge\tau_\epsilon^+}, \overline{X}_{t\wedge\tau_\epsilon^+}), \quad t \geq 0, \tag{7.6}
$$

is a right-continuous $\mathbb{P}_{x,s}$-supermartingale for all $(x,s) \in E$ such that $s < \epsilon$. Note that in this case $Z^{(q)} \in C^1(\mathbb{R}) \cap C^2(\mathbb{R}\setminus\{0\})$ and hence

$$
h_v(x) := e^{vx} Z_v^{(q-\psi(v))}(x)
$$

is in $C^1(\mathbb{R}) \cap C^2(\mathbb{R}\setminus\{0\})$, where $v \geq 0$. Now let $\Gamma$ be the infinitesimal generator of $X$ under $\mathbb{P}_0$ and formally define the function $\Gamma h_v : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ by

$$
\begin{aligned}
\Gamma h_v(x) := &-\gamma h_v'(x) + \frac{\sigma^2}{2} h_v''(x)\\
&+ \int_{(-\infty,0)} \big(h_v(x+y) - h_v(x) - y h_v'(x) 1_{\{y\geq -1\}}\big)\,\Pi(dy).
\end{aligned}
$$

The regularity of $h_v$ together with Taylor's theorem allows one to show that the quantity $\Gamma h_v(x)$ is well defined for $x > 0$. Moreover, for $x < 0$, we have $h_v(x) = e^{vx}$ and hence $\Gamma h_v(x)$ is well defined, too. Applying an appropriate version of the Itô–Meyer formula (cf. Theorem 71, Chap. IV of [21]) to $e^{-qt} h_v(X_t)$, we find that

$$
e^{-q(t\wedge\tau_0^-\wedge\tau_b^+)} h_v(X_{t\wedge\tau_0^-\wedge\tau_b^+}) - \int_0^{t\wedge\tau_0^-\wedge\tau_b^+} e^{-qu}\big(\Gamma h_v(X_u) - q h_v(X_u)\big)\,du, \quad t \geq 0,
$$

is a $\mathbb{P}_x$-martingale for $x \in (0,b)$. The martingale property of the first term (which is proved in the Appendix) then implies that

$$
\Gamma h_v(x) - q h_v(x) = 0, \quad x \in (0,b).
$$

Moreover, setting $\rho(y) := e^{vy}$, $y \in \mathbb{R}$, one may show that $\Gamma\rho(y) = \psi(v)\rho(y)$ for $y \in \mathbb{R}$ by taking Laplace transforms on both sides. Hence it follows for $x < 0$ that

$$\Gamma h_v(x) - qh_v(x) = \Gamma\rho(x) - \psi(v)\rho(x) - \big(q - \psi(v)\big)\rho(x) = -\big(q - \psi(v)\big)e^{vx}.$$

Next, fix $(x,s) \in E$ such that $x \le s < \epsilon$ and define the semimartingale $Y$ by $Y_t := X_t - \overline{X}_t + g_\epsilon(\overline{X}_t)$. We then have

$$\begin{aligned}
&e^{-q(t\wedge\tau_\epsilon^+)}V_\epsilon(X_{t\wedge\tau_\epsilon^+}, \overline{X}_{t\wedge\tau_\epsilon^+})\\
&= e^{-q(t\wedge\tau_\epsilon^+)}\Big(e^{\overline{X}_{t\wedge\tau_\epsilon^+}} h_0(Y_{t\wedge\tau_\epsilon^+}) - Ke^{\overline{X}_{t\wedge\tau_\epsilon^+} - g_\epsilon(\overline{X}_{t\wedge\tau_\epsilon^+})} h_1(Y_{t\wedge\tau_\epsilon^+})\Big).
\end{aligned}$$

Applying an appropriate version of the Itô–Meyer formula (cf. Theorem 71, Chap. IV of [21]) to $h_0(Y_{t\wedge\tau_\epsilon^+})$ and $h_1(Y_{t\wedge\tau_\epsilon^+})$ (see [11, 15] for a similar argument) and then using stochastic integration by parts for semimartingales (cf. Corollary 2 of Theorem 22, Chap. II of [21]), one obtains, $\mathbb{P}_{x,s}$-a.s.,

$$\begin{aligned}
&e^{-q(t\wedge\tau_\epsilon^+)}V_\epsilon(X_{t\wedge\tau_\epsilon^+}, \overline{X}_{t\wedge\tau_\epsilon^+})\\
&= V_\epsilon(x,s) + \tilde{M}_t\\
&\quad + \int_0^{t\wedge\tau_\epsilon^+} e^{-qu+\overline{X}_u}\\
&\qquad \times\Big(\Gamma h_0(Y_u) - qh_0(Y_u) - Ke^{-g_\epsilon(\overline{X}_u)}\big(\Gamma h_1(Y_u) - qh_1(Y_u)\big)\Big)\,du \qquad (7.7)\\
&\quad + \int_0^{t\wedge\tau_\epsilon^+} e^{-qu+\overline{X}_u}\Big(h_0(Y_u) + h_0'(Y_u)\big(g_\epsilon'(\overline{X}_u) - 1\big)\\
&\qquad\qquad -Ke^{-g_\epsilon(X_u)}\big(-h_1(Y_u) + h_1'(Y_u)\big)\big(g_\epsilon'(\overline{X}_u) - 1\big)\Big)\,d\overline{X}_u,
\end{aligned}$$

for some zero-mean martingale $\tilde{M}$ whose specific form is irrelevant. We claim that the first integral in (7.7) is a decreasing process. Indeed, we have $\Gamma h_0(x) - qh_0(x) = 0$ and $\Gamma h_1(x) - qh_1(x) = 0$ for $x > 0$. Moreover, we have $\Gamma h_0(x) - qh_0(x) = -q$ and $\Gamma h_1(x) - qh_1(x) = -(q - \psi(1))e^x$ for $x < 0$. Hence the first integrand is nonpositive provided that

$$-q + Ke^{-g_\epsilon(\overline{X}_t)}\big(q - \psi(1)\big)e^{Y_t} \le 0 \quad \text{on } \{Y_t \le 0\}.$$

This is clear if $q \le \psi(1)$. When $q > \psi(1)$, recall from (7.2) that $g_\epsilon(s) \ge \eta \vee 0$ and thus

$$-q + Ke^{-g_\epsilon(\overline{X}_t)}\big(q - \psi(1)\big)e^{Y_t} \le -q + Ke^{-(0\vee\eta)}\big(q - \psi(1)\big) \quad \text{on } \{Y_t \le 0\}.$$

Again due to the fact that $\beta_0 \vee 0 \le \eta \vee 0$, the right-hand side is smaller than zero and hence the first integral in (7.7) is a decreasing process.

The second integral in (7.7) vanishes since the process $\overline{X}$ only increases when $\overline{X}_u = X_u$ and by definition of $g_\epsilon$. Thus the process $e^{-q(t\wedge\tau_\epsilon^+)}V_\epsilon(X_{t\wedge\tau_\epsilon^+}, \overline{X}_{t\wedge\tau_\epsilon^+})$, $t \geq 0$, can be written as the sum of an initial value, a martingale and a decreasing process. Moreover, the decreasing process is of the form $\int_0^{t\wedge\tau_\epsilon^+} e^{-qu+\overline{X}_u}\tilde{f}(Y_u)\,du$, where $|\tilde{f}(Y_u)| \leq M$ for some constant $M$. Recall that $\tau_\epsilon^+ = \inf\{t > 0 : X_t > \epsilon\}$ so that $\overline{X}_u \leq \epsilon < \infty$ for $u \leq \tau_\epsilon^+$. Hence,

$$\int_0^{t\wedge\tau_\epsilon^+} e^{-qu+\overline{X}_u}|\tilde{f} f(Y_u)|\,du \leq e^\epsilon M \int_0^{t\wedge\tau_\epsilon^+} e^{-qu}\,du$$

implies that the decreasing process is integrable. Therefore, the process in (7.7) is a $\mathbb{P}_{x,s}$-supermartingale.

Finally, with all the preparation done, we can now prove the assertion, that is, show that the process $e^{-qt}V_\epsilon(X_t, \overline{X}_t)$, $t \geq 0$, is a right-continuous $\mathbb{P}_{x,s}$-supermartingale for $(x,s) \in E$. In view of Proposition 3.1, it suffices to assume that $(x,s) \in E$ such that $s < \epsilon$. Moreover, by the Markov property (see [11, 15] for a similar argument), it is enough to show that

$$\mathbb{E}_{x,s}[e^{-qt}V_\epsilon(X_t, \overline{X}_t)] \leq V_\epsilon(x,s).$$

Using the strong Markov property and Proposition 3.1, we now obtain

$$\begin{aligned}
\mathbb{E}_{x,s}\big[e^{-qt}V_\epsilon(X_t, \overline{X}_t)\big|\mathcal{F}_{\tau_\epsilon^+}\big] &= e^{-qt}V_\epsilon(X_t, \overline{X}_t)1_{\{t<\tau_\epsilon^+\}} \\
&\quad + E_{x,s}\big[e^{-qt}V_\epsilon(X_t, \overline{X}_t)\big|\mathcal{F}_{\tau_\epsilon^+}\big]1_{\{t\geq\tau_\epsilon^+\}} \\
&= e^{-qt}V_\epsilon(X_t, \overline{X}_t)1_{\{t<\tau_\epsilon^+\}} \\
&\quad + e^{-q\tau_\epsilon^+}E_{\epsilon,\epsilon}\big[e^{-qt}V_\epsilon(X_t, \overline{X}_t)\big]1_{\{t\geq\tau_\epsilon^+\}} \\
&\leq e^{-q(t\wedge\tau_\epsilon^+)}V_\epsilon(X_{t\wedge\tau_\epsilon^+}, \overline{X}_{t\wedge\tau_\epsilon^+}).
\end{aligned}$$

Taking expectations on both sides and using that the process in (7.6) is a $\mathbb{P}_{x,s}$- supermartingale, we get

$$\mathbb{E}_{x,s}[e^{-qt}V_\epsilon(X_t, \overline{X}_t)] \leq \mathbb{E}_{x,s}\big[e^{-q(t\wedge\tau_\epsilon^+)}V_\epsilon(X_{t\wedge\tau_\epsilon^+}, \overline{X}_{t\wedge\tau_\epsilon^+})\big] \leq V_\epsilon(x,s).$$

This completes the proof in the unbounded variation case.

*Bounded variation case:* If $X$ has bounded variation, then the Itô–Meyer formula is nothing more than an appropriate version of the change of variable formula for Stieltjes integrals, and the rest of the proof follows the same line of reasoning as above. The only change worth mentioning is that the generator of $X$ takes a different form. Specifically, one has to work with

$$\varGamma\tilde{f}(x) = \mathtt{d}\tilde{f}'(x) + \int_{(-\infty,0)}\big(\tilde{f}(x+y) - \tilde{f}(x)\big)\varPi(dy)$$

for appropriate $\tilde{f}$. □

*Proof of Lemma 7.3* The assertion is again true for $(x,s)\in E$ such that $s\geq\epsilon$ by Proposition 3.1. Thus, let $(x,s)\in E$ such that $s<\epsilon$. The assertion is clear if $x-s+g_\epsilon(s)\leq 0$. Hence, suppose that $s<\epsilon$ and $x-s+g_\epsilon(s)>0$. Replacing $t\wedge\tau_\epsilon^+$ by $t\wedge\tau_\epsilon^+\wedge\rho_\epsilon^*$ in (7.7) and recalling that $\Gamma h_0(y)=qh_0(y)$ and $\Gamma h_1(y)=qh_1(y)$ for $y>0$ shows that

$$\mathbb{E}_{x,s}\big[e^{-q(t\wedge\tau_\epsilon^+\wedge\rho_\epsilon^*)}V_\epsilon\big(X_{t\wedge\tau_\epsilon^+\wedge\rho_\epsilon^*},\overline{X}_{t\wedge\tau_\epsilon^+\wedge\rho_\epsilon^*}\big)\big]=V_\epsilon(x,s)$$

and hence by dominated convergence

$$\mathbb{E}_{x,s}\big[e^{-q(\tau_\epsilon^+\wedge\rho_\epsilon^*)}V_\epsilon\big(X_{\tau_\epsilon^+\wedge\rho_\epsilon^*},\overline{X}_{\tau_\epsilon^+\wedge\rho_\epsilon^*}\big)\big]=V_\epsilon(x,s). \tag{7.8}$$

Using the strong Markov property, one may now deduce that

$$\mathbb{E}_{x,s}\big[e^{-q\rho_\epsilon^*}V_\epsilon(X_{\rho_\epsilon^*},\overline{X}_{\rho_\epsilon^*})\big|\mathcal{F}_{\tau_\epsilon^+}\big]=e^{-q(\tau_\epsilon^+\wedge\rho_\epsilon^*)}V_\epsilon\big(X_{\tau_\epsilon^+\wedge\rho_\epsilon^*},\overline{X}_{\tau_\epsilon^+\wedge\rho_\epsilon^*}\big),$$

and thus taking expectations on both sides and using (7.8) gives the desired result. □

*Proof of Corollary 4.5* (i) Since $q>\psi(1)$, Lemma A.1 in the Appendix of [11] implies that

$$\mathbb{E}_{x,s}\Big[\sup_{0\leq t<\infty}e^{-qt}\big(e^{\overline{X}_t}-Ke^{X_t}\big)^+\Big]\leq\mathbb{E}_{x,s}\Big[\sup_{0\leq t<\infty}e^{-qt+\overline{X}_t}\Big]<\infty \tag{7.9}$$

for $(x,s)\in E$.

For $\epsilon\in\mathbb{R}$, let $V_\epsilon^*$, $\rho_\epsilon^*$ and $g_\epsilon$ be as in Theorem 4.4 and $V_\infty^*$ and $\rho_\infty^*$ as in Corollary 4.5. It follows by the construction of $g_\epsilon$ that $\lim_{\epsilon\uparrow\infty}g_\epsilon(s)=k^*\in[0,\infty)$ for $s\in\mathbb{R}$ which in turn implies that $\lim_{\epsilon\uparrow\infty}\rho_\epsilon^*=\rho_\infty^*$, $\mathbb{P}_{x,s}$-a.s., for all $(x,s)\in E$. Moreover, it is clear that $\lim_{\epsilon\uparrow\infty}V_\epsilon^*(x,s)=V_\infty^*(x,s)$ due to the continuity of scale functions. Next, we claim that

(i) $V_\infty^*(x,s)\geq(e^s-Ke^x)^+$ for $(x,s)\in E$;
(ii) $e^{-qt}V_\infty^*(X_t,\overline{X}_t)$, $t\geq 0$, is a $\mathbb{P}_{x,s}$-supermartingale for $(x,s)\in E$;
(iii) $V_\infty^*(x,s)=\mathbb{E}_{x,s}[e^{-q\rho_\infty^*}(e^{\overline{X}_{\rho_\infty^*}}-Ke^{X_{\rho_\infty^*}})^+]$ for $(x,s)\in E$.

Condition (i) is satisfied since $V_\epsilon^*(x,s)\geq(e^s-Ke^x)^+$ for $(x,s)\in E$ by Theorem 4.4 and the inequality remains valid in the limit. To prove (ii), use Fatou's lemma and Lemma 7.2 to show that

$$\begin{aligned}\mathbb{E}_{x,s}[e^{-qt}V_\infty^*(X_t,\overline{X}_t)]&\leq\liminf_{\epsilon\to\infty}E_{x,s}[e^{-qt}V_\epsilon^*(X_t,\overline{X}_t)]\\&\leq\liminf_{\epsilon\to\infty}V_\epsilon^*(x,s)\\&=V_\infty^*(x,s)\end{aligned}$$

for $(x,s)\in E$. By the Markov property, this inequality implies the desired $\mathbb{P}_{x,s}$-supermartingale property (see [11, 15] for a similar argument). As for (iii), using (7.9) and dominated convergence, we deduce that

$$\begin{aligned}
V_\infty^*(x,s) &= \lim_{\epsilon\to\infty} V_\epsilon^*(x,s)\\
&= \lim_{\epsilon\to\infty} \mathbb{E}_{x,s}\big[e^{-q\rho_\epsilon^*}\big(e^{\overline{X}_{\rho_\epsilon^*\wedge\epsilon}} - Ke^{X_{\rho_\epsilon^*}}\big)^+\big]\\
&= \mathbb{E}_{x,s}\big[e^{-q\rho_\infty^*}\big(e^{\overline{X}_{\rho_\infty^*}} - Ke^{X_{\rho_\infty^*}}\big)^+\big]
\end{aligned}$$

for $(x,s)\in E$. The proof of the corollary is now completed by using (i)–(iii) in the same way as in the proof of Theorem 4.4.

(ii) For $\epsilon\in\mathbb{R}$, let $V_\epsilon^*$, $\rho_\epsilon^*$ and $g_\epsilon$ be as in Theorem 4.4. It follows by the construction of $g_\epsilon$ that $\lim_{\epsilon\uparrow\infty} g_\epsilon(s)=\infty$ and $\lim_{\epsilon\uparrow\infty} V_\epsilon^*(x,s)=\infty$ for $x\le s$ and hence

$$V_\infty^*(x,s) \ge \lim_{\epsilon\uparrow\infty} \mathbb{E}_{x,s}\big[e^{-q\rho_\epsilon^*}\big(e^{\overline{X}_{\rho_\epsilon^*}} - Ke^{X_{\rho_\epsilon^*}}\big)\big] = \lim_{\epsilon\uparrow\infty} V_\epsilon^*(x,s) = \infty.$$

This completes the proof. □

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## Appendix: An auxiliary result

The goal of this section is to prove an auxiliary result that was used in the proof of Lemma 7.2. More precisely, for $q,v\ge 0$, we claim that the process

$$e^{-q(t\wedge\tau_0^-\wedge\tau_b^+)} h_v(X_{t\wedge\tau_0^-\wedge\tau_b^+}), \quad t\ge 0,$$

is a $\mathbb{P}_x$-martingale for $x\in(0,b)$. To see this, recall from Sect. 3.3 of [9] the identity

$$\begin{aligned}
f_1(x) &:= \mathbb{E}_x\Big[e^{-q\tau_0^- + vX_{\tau_0^-}} 1_{\{\tau_0^-<\infty\}}\Big]\\
&= e^{vx}\left(Z_v^{(q-\psi(v))}(x) - \frac{q-\psi(v)}{\Phi(q)-v} W_v^{(q-\psi(v))}(x)\right), \quad x\in\mathbb{R}. \qquad \text{(A.1)}
\end{aligned}$$

Applying the same technique (analytic extension) as in Sect. 3.3 of [9], one may also show that for $q,v\ge 0$ and $x\in(0,b)$,

$$f_2(x) := \mathbb{E}_x\Big[e^{-q\tau_b^+ + vX_{\tau_b^+}} 1_{\{\tau_b^+<\tau_0^-\}}\Big] = e^{vx}\frac{W_v^{(q-\psi(v))}(x)}{W_v^{(q-\psi(v))}(b)}.$$

An application of the Markov property together with (A.1) yields for $t\ge 0$ that

$$
\begin{aligned}
\mathbb{E}_x\left[e^{-q\tau_0^-} f_1(X_{\tau_0^-}) 1_{\{\tau_0^- < \infty\}} \middle| \mathcal{F}_t\right] &= e^{-q\tau_0^-} f_1(X_{\tau_0^-}) 1_{\{\tau_0^- < t\}} \\
&\quad + e^{-qt}\mathbb{E}_{X_t}\left[e^{-q\tau_0^-} f_1(X_{\tau_0^-}) 1_{\{\tau_0^- < \infty\}}\right] 1_{\{\tau_0^- > t\}} \\
&= e^{-q\tau_0^-} f_1(X_{\tau_0^-}) 1_{\{\tau_0^- < t\}} \\
&\quad + e^{-qt}\mathbb{E}_{X_t}\left[e^{-q\tau_0^- + vX_{\tau_0^-}} 1_{\{\tau_0^- < \infty\}}\right] 1_{\{\tau_0^- > t\}} \\
&= e^{-q(t\wedge\tau_0^-)} f_1(X_{t\wedge\tau_0^-}),
\end{aligned}
$$

which shows that the process $e^{-q(t\wedge\tau_0^-)} f_1(X_{t\wedge\tau_0^-})$, $t \geq 0$, is a $\mathbb{P}_x$-martingale for $x > 0$. By Doob's optimal stopping theorem, it therefore follows that the process $e^{-q(t\wedge\tau_0^-\wedge\tau_b^+)} f_1(X_{t\wedge\tau_0^-\wedge\tau_b^+})$, $t \geq 0$, is a $\mathbb{P}_x$-martingale for $x \in (0, b)$. A similar argument as above shows that $e^{-q(t\wedge\tau_0^-\wedge\tau_b^+)} f_2(X_{t\wedge\tau_0^-\wedge\tau_b^+})$, $t \geq 0$, is a $\mathbb{P}_x$-martingale for $x \in (0, b)$ as well, and hence appropriately combining the two martingales completes the proof.

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